MOTION OF A CONTINUOUS MEDIUM WITH AN INTERNAL ANGULAR VELOCITIES CORRELATION

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Equations are analyzed which yield a closer approximation to the inertial terms in the conventional equations of hydrodynamics by accounting for the angular velocities correlation in adjacent volume elements. Both an isotropic and an anisotropic fluid are considered.

1. Much attention is nowadays paid to the equation of mechanics for a medium whose motion cannot be described by specifying the displacements of discrete particles and whose motion includes, as an additional degree of freedom, rotation of discrete volume elements [1-4]. These equations differ radically from the conventional equations of mechanics for a continuous medium. In this study the author proposes a somewhat different approach to the problem, by formulating it in more classical terms. It is well known that the conventional equations of motion for a continuum are derived from the molecular-kinetics equation, with the mean-squared velocity in any volume element assumed equal to the mean-over-the-volume velocity squared. The entire energy not expended on translatory motion of the center of mass is converted into internal thermodynamic energy. If the medium also rotates nonuniformly near a vortex filament, for example, then it becomes possible to introduce into the equations additional nondissipative terms accounting for regular motion. Such a correction may be appreciable in the case of highly associated fluids or solutions containing long linear chains of polymer molecules.

It has been stated in [4] that a system of equations is incomplete if it does not include rotation of a medium as the additional degree of freedom independent of forward displacements. Possibly, the authors of [4] have not taken into account the inertial terms associated with rotation. Here we will derive the equations of motion from the Lagrange function, thus ensuring completeness of the system and all necessary characteristics of invariability. As to the additional internal rotation of molecules, we do not yet quite know to what extent it can be accounted for in nonquantum terms. A transition from the rotation of discrete molecules to the concepts of continuum theory is yet to be validated even in classical mechanics.

2. The derivation does not require any special assumptions concerning the model characteristics. The medium retains its complete homogeneity at all points and its isotropy in all directions. The refinement here consists in expressing the kinetic energy not only in terms of the mean velocity squared but also in terms of the velocity curl squared, whose coefficient is proportional to the correlation distance between velocities. If many more than one molecule are packed into this distance, then the mechanics of continua is applicable. In this way, we have assumed that it is easier to rotate than to deform a fluid. In the general case it would be necessary also to include a term which is proportional to the deformation rate squared.

If dissipative forces are neglected, then Pascal's Law may be applied to a fluid, i.e., one may consider pressure to be a scalar quantity. Assuming the fluid to be incompressible, we will write for it the Lagrange function:

$$L \equiv \int \mathcal{L} dV = \int \left(\rho \frac{v^2}{2} + \frac{J}{2} (\mathrm{curl}\, \vec{v})^2 - p \right) dV. \tag{1}$$

In order to perform variational calculus here, we first change to Lagrangian coordinates, i.e.,

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$$\left(\frac{\partial\vec{v}}{\partial t}\right)_a=\left(\frac{\partial\vec{v}}{\partial t}\right)_r+(\vec{v}\nabla)\vec{v}.$$

Variation with respect to the curl $\vec{v}$ must account for the fact that a curl implies differentiation with respect to Eulerian and not Langrangian coordinates. Considering that the Jacobian of the transformation is equal to unity for an incompressible fluid, we write one of the partial derivatives as follows:

$$\frac{\partial v_2}{\partial x_1}=\frac{\partial(v_2,\ x_2,\ x_3)}{\partial(a_1,\ a_2,\ a_3)}\Big/\frac{\partial(x_1,\ x_2,\ x_3)}{\partial(a_1,\ a_2,\ a_3)}=\frac{\partial(v_2,\ x_2,\ x_3)}{\partial(a_1,\ a_2,\ a_3)}.$$

The variation of this expression is

$$\delta\ \frac{\partial v_2}{\partial x_1}=\frac{\partial(\delta x_2,\ x_2,\ x_3)}{\partial(a_1,\ a_2,\ a_3)}+\frac{\partial(v_2,\ \delta x_2,\ x_3)}{\partial(a_1,\ a_2,\ a_3)}+\frac{\partial(v_2,\ x_2,\ \delta x_3)}{\partial(a_1,\ a_2,\ a_3)}$$

$$=\frac{\partial\delta v_2}{\partial x_1}+\frac{\partial(v_2,\ \delta x_2)}{\partial(x_1,\ x_2)}+\frac{\partial(v_2,\ \delta x_3)}{\partial(x_1,\ x_2)}.$$

Here the first term is transformed, as usually, by parts of the expression for the force $S=\int L dt$. Integrating by parts the other two terms with respect to space, and changing to vector notation, we find the variational derivative of the Lagrange function:

$$\frac{\delta\mathcal{L}}{\delta r}=\rho\left(\frac{\partial\vec{v}}{\partial t}+(\vec{v}\nabla)\vec{v}\right)+J\left(\frac{\partial}{\partial t}\text{curl curl}\vec{v}+(\vec{v}\nabla)\text{curl curl}\vec{v}\right.$$

$$\left.+(\text{curl}\,\vec{v}\nabla)\text{curl}\vec{v}+(\text{curl curl}\vec{v}\nabla)\vec{v}\right)+\nabla p=0, \tag{2}$$

i.e., the equation of motion.

At a rigid wall it is necessary to let the tangential component of $\text{curl}_t\vec{v}$ vanish, since a wall does not allow a fluid to revolve about an axis in the plane of the wall. Furthermore, as usual, the normal component of velocity is also equal to zero here.

The normal components of the momentum flux vanish at the free surface. We define these components according to Eq. (2), which will be written to reflect the Conservation Law:

$$\frac{\partial}{\partial t}(\rho v_i+J\text{curl}_i\text{curl}\vec{v})+\frac{\partial}{\partial x_k}(p\delta_{ik}+\rho v_i v_k+Jv_i\text{curl}_k\text{curl}\vec{v}$$

$$+Jv_k\text{curl}_i\text{curl}\vec{v}+J\text{curl}_i\,\vec{v}\text{curl}_k\,\vec{v})=0. \tag{3}$$

Thus, the momentum density becomes

$$\rho v_i+J\text{curl}_i\text{curl}\vec{v}, \tag{4}$$

and the momentum flux density

$$p\delta_{ik}+\rho v_i v_k+J(v_i\text{curl}_k\text{curl}\,\vec{v}+v_k\text{curl}_i\,\text{curl}\,\vec{v}+\text{curl}_i\,\vec{v}\text{curl}_k\,\vec{v}). \tag{5}$$

The terms which are functions of the velocity and the velocity curl may be characterized as dynamic stresses. This tensor is symmetric. Asymmetry appears only when rotation not included in curl $\vec{v}$ is taken into account.

In order to find the moment density, one must vary the coordinates and the velocities corresponding to a rotation through a small angle $\delta\vec{\varphi}$. It is easy to show then that the variation $\delta\mathcal{L}$, which should vanish, is equal to

$$\delta\vec{\varphi}\ \frac{d}{dt}\ [\vec{r},\ \rho v+J\text{curl curl}\vec{v}]. \tag{6}$$

Consequently, the moment density is

$$[\vec{r},\ \rho\vec{v}+J\text{curl curl}\vec{v}], \tag{7}$$

in agreement with expression (5) for the momentum density.

In order to find the expressions for the energy density and the energy flux density, it is necessary, as usually, to multiply Eq. (2) scalarly by $\vec{v}$ and then integrate by parts over the volume. Except when the fluid rotates as a rigid body, one may let curve $\vec{v} = 0$ at an infinitely far surface. Then, the following expression is obtained for the energy density (retaining only the term which is a function of J):

$$E_J = \frac{J}{2} (\mathrm{curl}\, \vec{v})^2, \tag{8}$$

and for the energy flux density corresponding to (8):

$$J\vec{v}\left(\vec{v}\,\mathrm{curl}\ \mathrm{curl}\,\vec{v} + \frac{1}{2}(\mathrm{curl}\,\vec{v})^2\right). \tag{9}$$

We will now show that, when the entire fluid rotates as a rigid body, the energy contains no components proportional to curl $\vec{v}$. They drop out of Eq. (2), since in the case of a rotating rigid body curl $\vec{v}$ is not a function of the space coordinates. Expression (8) for the energy density cannot be separated now from the surface-integral components, since these do not vanish. With the substitution $\vec{v} = [\vec{\omega}\vec{r}]$, $d\vec{v}/dt = [\vec{\omega}\vec{r}]$ and with the surface integrals retained, all terms containing curl $\vec{v}$ are identically reducible.

In conclusion, we will derive an equation for the velocity distribution near a vortex filament. By analogy to conventional hydrodynamics but taking into account the angular velocities correlation, we have

$$\rho\,\mathrm{curl}\,\vec{v} + J\,\mathrm{curl}\,\mathrm{curl}\,\mathrm{curl}\,\vec{v} = 0. \tag{10}$$

From here we find the expression for the velocity

$$v = \frac{1}{r}\left(1 - r\sqrt{\frac{\rho}{J}}\,K_1\left(r\sqrt{\frac{\rho}{J}}\right)\right). \tag{11}$$

The filament radius is of the order of the quantity $\sqrt{J/\rho}$, i.e., is comparable to the correlation distance.

3. The motion of an anisotropic fluid or a fluid just produced by melting a single crystal may have anisotropic inertial properties. The inertia must now be described by a symmetric tensor of second rank rather than by the scalar J. The principal axes of this tensor must be thought of as rigidly coupled to a fluid particle, then it can be treated as some set of certain constants characterizing the fluid.

Since generally a volume element of fluid rotates continuously, the kinematics of rotation must be introduced into the equation of motion. There is a method known in which rotations of axes can be represented by four Euler parameters [5] with only one additional condition. These parameters are expressed by Euler angles as follows:

$$\begin{aligned} \xi_1 &= \sin\frac{\vartheta}{2}\sin\frac{\psi-\varphi}{2}; \quad \xi_2 = \sin\frac{\vartheta}{2}\cos\frac{\psi-\varphi}{2}; \\ \xi_3 &= \cos\frac{\vartheta}{2}\sin\frac{\psi+\varphi}{2}; \quad \xi_4 = \cos\frac{\vartheta}{2}\cos\frac{\psi+\varphi}{2}, \end{aligned} \tag{12}$$

where $\xi_\alpha \xi_\alpha = 1$. A double Greek symbol denotes summation from 1 to 4.

The angular velocities and the cosines of the angles between moving and fixed axes will be expressed in terms of parameters ξ_α and their time derivatives. In the conventional notation these equations are not quite symmetrical and, therefore, they can be formulated on the basis of certain general rules. An application of Pauli matrices [6] and of quarternions will be very helpful here.

For an arbitrary pair of quantities ξ_1, ξ_2 or ξ_3, ξ_4 the Pauli matrices are set up as follows:

$$\sigma_1 = \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}; \quad \sigma_2 = \begin{matrix} 0 & -i \\ i & 0 \end{matrix}; \quad \sigma_3 = \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix}. \tag{13}$$

The rows and the columns of these matrices are numbered with Greek letters equivalent to either 1, 2 or 3, 4. According to P. A. Dirac [6], we introduce another three matrices ρ_1, ρ_2, and ρ_3 which look like σ_1, σ_2, and σ_3 but act directly on the pair ξ_1, ξ_2 or ξ_3, ξ_4. In other words, ρ_1 singly represent both pairs, etc. Obviously, σ_i and ρ_k are transposable for any i and k, since they act on different variables. The following relations are valid within each triplet:

$$\begin{aligned} &\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 1; \quad \sigma_1\sigma_2 = -\sigma_2\sigma_1 = i\sigma_3; \\ &\sigma_3\sigma_1 = -\sigma_1\sigma_3 = i\sigma_2; \quad \sigma_2\sigma_3 = -\sigma_3\sigma_2 = i\sigma_1 \end{aligned} \tag{14}$$

and analogously for ρ_k.

We will now determine such operators called quarternions:

$$\Gamma_1 \equiv -i\rho_2\sigma_1;\ \Gamma_2 \equiv i\rho_2\sigma_3;\ \Gamma_3 \equiv -i\sigma_3. \tag{15}$$

Obviously, each operator $\Gamma_{1,2,3}$ is a matrix of fourth rank acting on the basis ξ_α. At the same time, however, the quantities denoted with symbols 1, 2 and 3, 4 have the vector property. Namely, during a rotation of the coordinate system Γ_1, Γ_2, and Γ_3 are symbolically transformed like vector components:

$$\Gamma_i' = \alpha_{i_k}\Gamma_k. \tag{16}$$

It can be easily proved with the aid of equalities (14) that the transform Γ_i' satisfies the same conditions as the original, i.e., that

$$\Gamma_1\Gamma_2 = -\Gamma_2\Gamma_1 = \Gamma_3;\ \Gamma_1^2 = \Gamma_2^2 = \Gamma_3^2 = 1 \text{ etc.} \tag{17}$$

With the aid of quarternions it becomes easy to express the projections of angular velocity on the moving system of coordinates:

$$\omega_i = \xi_\alpha (\Gamma_i)_{\alpha\beta}\, \dot{\xi}_\beta. \tag{18}$$

Omitting the Greek subscript, we will simply write

$$\omega_i = \xi\Gamma_i\dot{\xi}. \tag{18a}$$

As it turns out, the cosines α_{ik} themselves can be expressed in terms of quarternions. For this purpose one needs still another triplet of quarternions:

$$\overline{\Gamma}_1 \equiv i\rho_1\sigma_2;\ \overline{\Gamma}_2 \equiv i\rho_2;\ \overline{\Gamma}_3 = -i\rho_3\sigma_2. \tag{19}$$

$\overline{\Gamma}_1$, $\overline{\Gamma}_2$, and $\overline{\Gamma}_3$ are subject to the same rules of multiplication as Γ_1, Γ_2, and Γ_3, and all $\overline{\Gamma}_i$ quarternions are transposable with all Γ_k quarternions. The cosines of angles betwen the old and the new axes are expressed in terms of $\overline{\Gamma}_i$, Γ_k as follows:

$$\alpha_{ik} = \xi\overline{\Gamma}_i\Gamma_k\xi, \tag{20}$$

which allows us to use the tensor notation.

We will consider the rotation of some hypermolecular association or cluster to be entirely due to the motion of the volume element within which it finds itself. Such a motion resembles the rolling of a rigid body without sliding or the revolution of the moon around the earth: the displacement rate determines the angular velocity about the axis. In the case of a continuum, the projections of the angular velocities onto the fixed axes can be expressed in terms of curl $\vec{v}$:

$$\omega_i^0 = \frac{1}{2}\operatorname{curl}_i v. \tag{21}$$

In order to calculate the kinetic energy of rotation in the case of an anisotropic medium, one must project ω^0 onto the axes coupled to the medium. This is accomplished with the aid of relations (20):

$$\xi\Gamma_i\dot{\xi} = \omega_i = \alpha_{ik}\omega_k^0 = \xi\overline{\Gamma}_i\Gamma_k\xi \cdot \frac{1}{2}\operatorname{curl}_k \vec{v}. \tag{22}$$

Equations (22) yield the kinematic conditions for determining the Euler parameters ξ_α, if the relation $\xi\xi = \xi_\alpha\xi_\alpha = 1$ is also used. The derivative $\dot{\xi}_\alpha$ must be understood in the material sense.

During the motion of a continuum with anisotropic properties, however, the velocity cannot in fact be found separately from the parameters ξ_α also contained in the equations of motion. In order to write out these equations, we will construct the corresponding Lagrange function. In the absence of dissipative forces, the pressure must be first treated as a scalar quantity. Denoting the first principal moments of inertia by $J^{(1)}$, $J^{(2)}$, and $J^{(3)}$, we write the Lagrange function as follows:

$$L = \int \mathcal{L}\, dV = \int \left(\frac{\rho v^2}{2} + \frac{1}{2}\sum_{i=1}^{3} J^{(i)}\, \alpha_{ik}\alpha_{il} \operatorname{curl}_k \vec{v} \operatorname{curl}_l \vec{v} - p\right) dV. \tag{23}$$

In performing a variation, one must consider the variable quantities α_{ik} to be functions of the Lagrangian coordinates, as also the parameters ξ_α. The rotation field of local axes of inertia is then determined from the kinematic relations (22) and (23) together with the dynamic equations. The variation is calculated analogously as for the isotropic case. Let $\mathcal{L}_J$ be the component of the density of the Lagrange function which is due to rotation. Let

$$\delta\mathcal{L}_J \equiv \delta\,\frac{1}{2}\sum_{i=1}^{3} J^{(i)}\,\alpha_{ik}\alpha_{il}\operatorname{curl}_k\vec{v}\cdot\operatorname{curl}_l\vec{v} \equiv f_l\delta\operatorname{curl}_l v$$
$$= \frac{1}{2}\,f_{ik}\delta\left(\frac{\partial v_k}{\partial x_i}\cdot\frac{\partial v_i}{\partial x_k}\right), \tag{24}$$

where $f_l = (1/2)\varepsilon_{lik}f_{ik}$. Then the variational derivative is

$$\frac{\delta\mathcal{L}_J}{\delta x_l} = \frac{\partial f_{ik}}{\partial x_k}\cdot\frac{\partial v_l}{\partial x_k} + \operatorname{curl}_k\vec{v}\,\frac{\partial f_l}{\partial x_k} + \left(\frac{\partial}{\partial t} + (\vec{v}\nabla)\right)\frac{\partial f_{kl}}{\partial x_k}\,. \tag{25}$$

In this form it is not equal to zero, however, because of the nonholonomic coupling between the variables based on equality (22). In order to write the equation of motion, one must take this coupling into account as an additional condition. By virtue of the homogeneity of (22) with respect to the generalized velocities, properties which make the system conservative are retained, i.e., a nonholonomic coupling of this kind does not imply dissipation. It is to be noted that the equation does not contain second derivatives of ξ and, therefore, the parameters ξ are purely kinematic ones.

NOTATION

$\vec{a}\,(a_1, a_2, a_3)$	is the Lagrangian coordinates of a fluid particle;
α_{ik}	is the directional cosines of the angles between fixed and moving axes;
Γ_i	is the quarternions;
δ_{ik}	is the unit tensor;
ε_{ikl}	is the completely antisymmetric tensor;
$\varepsilon_{123} = -\varepsilon_{213} = 1$	
J	is the scalar coefficient characterizing the density of the moment of inertia;
$J^{(1)}, J^{(2)}, J^{(3)}$	is the density of the moments of inertia tensor (of its principal values);
K_1	is the first-order MacDonald function;
L	is the Lagrange function for a fluid;
$\mathcal{L}$	is the density of the Lagrange function for a fluid;
ω_i	is the components of the angular velocity vector;
p	is the pressure;
$\vec{r}\,(x_1, x_2, x_3)$	if the radius vector in the fixed system of coordinates;
ρ	is the density of the medium;
ρ_1, ρ_2, ρ_3	is the matrices analogous to the Pauli matrices;
S	is the force;
$\sigma_1, \sigma_2, \sigma_3$	is the Pauli matrices;
$v(v_1, v_2, v_3)$	is the velocity of the fluid;
$\xi_\alpha\,(\alpha = 1, 2, 3, 4)$	is the Euler parameters;
φ, ψ, ϑ	is the Euler angles.

LITERATURE CITED

1. I. S. Dahler, J. Chem. Phys., 30, 1447 (1959).
2. E. L. Aero and E. V. Kuvshinskii, Fiz. Tverd. Tela, 2, 1399 (1960).
3. E. L. Aero, A. N. Bulygin, and E. V. Kuvshinskii, Prikl. Matem. i Mekhan., 29, 297 (1965).
4. E. V. Kuvshinskii and E. L. Aero, Fiz. Tverd. Tela, 5, 2591 (1963).
5. E. T. Whittacker, Analytical Dynamics [Russian translation], Moscow (1937), p. 24.
6. P. A. M. Dirac, Fundamentals of Quantum Mechanics [Russian translation], Moscow (1937), p. 271.